Indian Statistical Institute
First Semester Exam, 2006-2007
B.Math II Year
Analysis III

Time: 3 hrs Date:29-11-06

Note: Maximum marks you can get is 50 out of a total of 51 marks.

1. Let $g : (a, b) \to R$ be a C^2 function i.e, g, g', g'' are all continuous. Let $g'(x_0) = 0$ and $g''(x_0) \neq 0$ for some x_0 in (a, b). For each $\delta > 0$ show that there exist p_1, p_2 in $(x_0 - \delta,\ x_0 + \delta), p_1 \neq p_2$ such that $g(p_1) = g(p_2)$. [3]

2. Let $f : R \to R$ be given by
   $f(t) = t + 2t^2 \sin(\frac{1}{t})$ for $t \neq 0$
   $f(0) = 0.$

   (a) Show that $f'(0) = 1, f'$ is not continuous at 0. [1]

   (b) For each $\delta > 0$ show that f is not $1 - 1$ on $(0, \delta)$. [3]

   (Use problem 1 if necessary).

3. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

   can be solved for

$$\begin{array}{ll} x, y, u & \text{in terms of } z \\ x, z, u & \text{in terms of } y \\ y, z, u & \text{in terms of } x \end{array}$$

   but not for x, y, z in terms of u. [4]

4. Let $a, b > 0, \quad c < 0$. Assume that g, h, f are reals. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse. Find the area enclosed by it in terms of a, b, c, g, h, f. [4]

5. Show that

$$\iint\limits_{x_1^2+x_2^2+\ldots+x_n^2<1} \cdots \int x_k^2 dx_1 dx_2 \ldots dx_n = \frac{1}{n+2} \iint\limits_{x_1^2+x_2^2+\ldots+x_n^2<1} \cdots \int dx_1 \ldots dx_n$$

for each $k = 1, 2, \ldots n$. [Hint: You can assume that there exists intervals $I_1, I_2, \ldots, I_{n-1}$ and smooth functions $f_1, f_2, \ldots, f_{n-1}$ of $\theta_1, \theta_2, \ldots, \theta_{n-1}$ such that $\lambda : (0, \infty) \times I_1 \times I_2 \times \ldots \times I_{n-1} \rightarrow R^n \backslash \{0\}$ given by $\lambda(r, \theta_1, \ldots, \theta_{n-1}) = (x_1, x_2, \ldots, x_n), x_j = r f_j(\theta_1, \ldots, \theta_{n-1})$ is $1-1$, onto].

[4]

6. Show that $\lambda : (0, \infty) \times [0, 2\pi) \times [0, \pi] \times [0, \pi] \rightarrow R^4 \backslash 0$ given by

$$\begin{aligned}
\lambda(r, \theta_1, \theta_2, \theta_3) &= (x_1, x_2, x_3, x_4) \\
x_4 &= r \cos \theta_3 \\
x_3 &= r \sin \theta_3 \cos \theta_2 \\
x_2 &= r \sin \theta_3 \sin \theta_2 \cos \theta_1 \\
x_1 &= r \sin \theta_3 \sin \theta_2 \sin \theta_1
\end{aligned}$$

is $1-1$, onto. [4]

7. Let $-\infty < a_1 < b_1 < a_2 < b_2 < \infty$. Let $f_1 : [a_1, b_1] \rightarrow R$, $f_2 : [a_2, b_2] \rightarrow R$ be continuous functions. Given $\epsilon > 0$ show that there exists *one* polynomial P such that $\sup\limits_{a_j \leq x \leq b_j} |P(x) - f_j(x)| \leq \epsilon$ for *each* $j = 1, 2$.

[3]

8. Let $f : [a, b] \rightarrow R$ be continuous. Given $\epsilon > 0$ show that there exists a polynomial P with rational coefficients such that $\sup\limits_{a \leq x \leq b} |f(x) - P(x)| \leq \epsilon$.

[3]

9. (a) For any rational number $x \neq 0, x = \frac{a}{b}, \quad a, b$ integer, $b > 0, gcd(a,b) = 1$, define $\lambda(x) = a$. Show that for any interval J of positive length,

$$\sup\{|\lambda(x)| : \ x \text{ rational }, x \in J, x \neq 0\} = \infty$$

[2]

(b) Let $f_n, g_n : R \to R$ be given by

$$\begin{aligned} f_n(x) &= x\left(1+\frac{1}{n}\right), \\ g_n(x) &= \begin{cases} \frac{1}{n} & \text{if } \quad x = 0 \text{ or } x \text{ is irrational} \\ b + \frac{1}{n} & \text{if } \quad x \text{ is rational } x = \frac{a}{b}, \\ & b > 0, a, b \text{ integers, } \ gcd(a,b) = 1. \end{cases} \end{aligned}$$

Let $h_n(x) = f_n(x) g_n(x)$

(i) Show that f_n and g_n converge uniformly on each bounded interval. [1]

(ii) Using part (a) if necessary, show that h_n does not converge uniformly on any interval J of positive length. [3]

10. Use divergence theorem to evaluate $\iint_S \underset{\sim}{F}$ where $\underset{\sim}{F}(x,y,z) = (xy^4, x^4y, 2x^2y^2z)$ and S is the surface oriented outwards of the solid cylinder $x^2 + y^2 \leq 100$ bounded above by the plane $x + z = 0$ and below by the plane $z = 0$. [4]

11. For smooth vector fields $\underset{\sim}{K}, \underset{\sim}{L} : R^3 \to R^3$ and smooth scalar field $f : R^3 \to R$ show that

(a) div $(\underset{\sim}{K} \times \underset{\sim}{L}) = (\text{curl } \underset{\sim}{K}) \cdot \underset{\sim}{L} - \underset{\sim}{K} \cdot \text{ curl } \underset{\sim}{L}$.

[3]

(b) curl $(f\underset{\sim}{K}) = (\nabla f) \times \underset{\sim}{K} + f \text{ curl } \underset{\sim}{K}$ [3]

12. Let S denote the surface of intersection of the plane $x + y = 2b$ and the solid sphere $x^2 + y^2 + z^2 \leq 2b(x+y)$. Let Γ be the boundary of S. Let $\underset{\sim}{F}(x,y,z) = (y,z,x)$. Verify Stokes theorem for $\underset{\sim}{F}, S, \Gamma$. [6]